Exercise 10

Use Exercises 6 and 8 and superposition to solve the wave equation with initial data

$$u(x,0) = e^{-x^2}, \quad \frac{\partial u}{\partial t}(x,0) = \frac{x}{(1+x^2)^2}, \quad -\infty < x < \infty.$$

Solution

The aim is to solve the wave equation on the whole line for all time subject to initial conditions.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ -\infty < t < \infty$$

$$u(x,0) = e^{-x^2}$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{x}{(1+x^2)^2}$$

Take advantage of the fact that the wave equation is a linear equation by setting $u(x,t) = v(x,t) + w(x,t)$. The PDE becomes

$$\frac{\partial^2}{\partial t^2}(v+w) = c^2 \frac{\partial^2}{\partial x^2}(v+w)$$

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 w}{\partial t^2} = c^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \right)$$

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} + c^2 \frac{\partial^2 w}{\partial x^2}.$$

For this equation to remain satisfied, set

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

On the other hand, the initial conditions become

$$u(x,0) = v(x,0) + w(x,0) = e^{-x^2}$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{\partial v}{\partial t}(x,0) + \frac{\partial w}{\partial t}(x,0) = \frac{x}{(1+x^2)^2}.$$

For the conditions to remain satisfied, set

$$v(x,0) = e^{-x^2} \qquad\qquad w(x,0) = 0$$

$$\frac{\partial v}{\partial t}(x,0) = 0 \qquad\qquad \frac{\partial w}{\partial t}(x,0) = \frac{x}{(1+x^2)^2}.$$

To summarize, the initial value problem for u is equivalent to the following problems for v and w,

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}, \quad -\infty < x, t < \infty \qquad \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \quad -\infty < x, t < \infty$$

$$v(x,0) = e^{-x^2} \qquad w(x,0) = 0$$

$$\frac{\partial v}{\partial t}(x,0) = 0 \qquad \frac{\partial w}{\partial t}(x,0) = \frac{x}{(1+x^2)^2},$$

which have been solved for already in Exercise 6 and Exercise 8, respectively.

$$v(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right] \qquad w(x,t) = \frac{1}{4c}\left[\frac{1}{1+(x-ct)^2} - \frac{1}{1+(x+ct)^2}\right]$$

Therefore, since $u(x,t) = v(x,t) + w(x,t)$,

$$u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right] + \frac{1}{4c}\left[\frac{1}{1+(x-ct)^2} - \frac{1}{1+(x+ct)^2}\right].$$

Below are plots of $u(x,t)$ versus x over $-20 < x < 20$ for $t = 0, 1, 2, 4, 6, 8$ with $c = 1$.